A Dynamic Analysis of Speculation Across Two Markets

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Abstract

A discrete time model of a financial market is proposed, where the time evolution of asset prices and wealth arises from the interaction of two groups of agents, fundamentalists and chartists. Each group allocates its wealth between a risky asset (stock) and an alternative asset (bond), and the two groups have heterogeneous expectations about returns. We assume that chartists compute expected returns by extrapolating past price changes, while fundamentalists form their expectations on the basis of their superior knowledge of fundamentals. Under the assumption that agents have CRRA utility, investors' optimal demand for each asset depends on their wealth, and this results in growing price and wealth processes. The time evolution of the prices is modeled by assuming the existence of a market maker, who sets excess demand of each asset to zero at the end of each trading period by taking an off-setting long or short position. The market maker is assumed to adjust the price, in each period, partly on the basis of the excess demand and partly according to a particular market stabilization policy. The model is reduced to a high dimensional nonlinear discrete-time dynamical system with growing prices and wealth. Although the model is nonstationary, suitable changes of variables lead to a stationary model where the dynamic variables are actual and expected returns, fundamental/price ratios, and wealth proportions of chartists and fundamentalists. The steady states and other invariant sets of the model are determined, and important global dynamic phenomena are studied via numerical techniques. Stochastic simulations are also performed, that show the ability of the model to generate some of the characteristic features of financial time series.

1 Introduction

In recent years several models of asset price dynamics based on the interaction of *heterogeneous agents* have been proposed (Day and Huang (1990), Brock and Hommes (1998), Lux (1998), Chen and Yeh (1997), Gaunersdorfer (2000), Chiarella and He (2001, 2002a,b), Fernandez-Rodriguez et al. (2002)). Most of these models, some of which allow the size of the different groups of agents to vary according to the evolution of the financial market, are of necessity not very mathematically tractable. In a previous paper, Chiarella, Dieci and Gardini (2002), whose antecedents are Chiarella (1992), Beja and Goldman (1980), and Zeeman (1974), we developed a two-dimensional discrete time model of asset price dynamics containing the essential elements of the heterogeneous agents paradigm whilst still remaining mathematically tractable. In that model we assumed a financial market with a risky asset and an alternative riskless asset, consisting of two types of traders: fundamentalists, holding a superior knowledge of fundamentals, and chartists, basing their trading decisions on an analysis of past price trends. However, the model studied in Chiarella, Dieci and Gardini (2002) is a partial one since it leaves in the background the dynamics of the market for the alternative asset. This results in the equilibrium not being at the fundamental value of the risky asset. Moreover, in that model agents' optimal demands for the risky asset are independent of their wealth, as a result of the underlying CARA utility functions, and therefore the time evolution of agents' wealth has no effect on price dynamics.

In the present paper, again assuming a financial market consisting of fundamentalists and chartists, we develop a more complete model where the dynamics of the price of the alternative asset, and its dependence on agents' investment decisions, is also taken into account. This avoids the unintuitive steady state of the earlier model. Moreover, we consider a more realistic framework where investors' optimal decisions depend on their wealth (as a result of underlying CRRA utility functions) and both price and wealth processes are thus growing. Each group forms expectations about asset returns and allocates its wealth between the risky asset and the alternative asset. The time evolution of the prices is modelled by assuming the existence of a *market maker*, who sets excess demand to zero at the end of each trading period by taking an off-setting long or short position. The market maker also decides the next period prices partly on the basis of the excess demand (because of inventory reasons) and partly in order to steer prices back to their equilibrium (fundamental) values (to ensure an "orderly" adjustment of prices).

The model is reduced to a high-dimensional nonlinear discrete-time dynamical system that describes the time evolution of actual returns, agents' beliefs about expected returns, fundamental/price ratios and wealth proportions of the two groups. Despite the high dimension of the model, analytical results can be obtained in some particular lower-dimensional cases, and these help to understand the global dynamic behaviour of the dynamical system.

The structure of the paper is as follows. Section 2 derives the asset demand functions for each asset by each investor type. Section 3 describes how demands are aggregated by the market maker via a price adjustment rule in the market for each asset. Section 4 describes the resulting dynamical system for the time evolution of actual and expected returns, fundamental/price ratios and wealth proportions. Section 5 focuses on the equilibria of the model, outlining some results about their local asymptotic stability, and analyzes the restriction of the dynamics of the model to some important invariant subsets of the phase-space. Using numerical and graphical tools, Section 6 performs some global analysis of the out-of-equilibrium dynamics of the model, focusing on phenomena of regular or chaotic oscillatory dynamics, intermittent behaviour, and coexistence of attracting sets in the phase space. Section 7 performs some stochastic simulations in order to show how the interaction of the nonlinear deterministic phenomena of the model with simple noise processes can give rise to some of the typical distributional features of asset returns.

2 Asset demand and expectation formation

We label with 1 the risky asset (stock) and with 2 the alternative asset (bond). For $i = 1, 2$ we denote by $P_{i,t}$ the price of the i-th asset at time t, by g_i the dividend (or coupon) yield, assumed constant, produced by the i-th asset from t to $t+1$; we use the subscript $j \in \{f, c\}$ to denote fundamentalists or chartists. In each time period each group of agents is assumed to invest some of its wealth in the risky asset and some in the risk-free asset. Denote by $\Omega_t^{(j)}$ and $Z_{i,t}^{(j)}$ the wealth of agent j and the fraction of that wealth that agent j decides to invest in the i-th asset at time t, respectively, with $Z^{(j)}_{1,t}+Z^{(j)}_{2,t}=1$. The time evolution of the wealth of agent j satisfies

$$\begin{aligned}\Omega^{(j)}_{t+1} &= \Omega^{(j)}_t+\Omega^{(j)}_t Z^{(j)}_{1,t}(\frac{P_{1,t+1}-P_{1,t}}{P_{1,t}}+g_1)\\ &\quad +\Omega^{(j)}_t(1-Z^{(j)}_{1,t})(\frac{P_{2,t+1}-P_{2,t}}{P_{2,t}}+g_2)\ , \qquad (1)\end{aligned}$$

where $(P_{i,t+1}-P_{i,t})/P_{i,t}$ is the capital gain on the i-th asset over $(t,t+1)$.

We denote by $E^{(j)}_t$, $Var^{(j)}_t$ the "beliefs" of investor type j, at time t, about conditional expectation and variance, respectively. It is assumed that all the investors have the same attitude to risk with the same CRRA utility of wealth function[1] $u(\Omega)=\log(\Omega)$. Agent j seeks the fractions $Z^{(j)}_{i,t}$, $i=1,2$, so as to maximize $E^{(j)}_t[\log(\Omega^{(j)}_{t+1})]$, the expected utility of wealth at time $t+1$.

For $j\in\{f,c\}$, $i=1,2$, denote by

$$m^{(j)}_{i,t}=E^{(j)}_t\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\ , \qquad V^{(j)}_{i,t}=Var^{(j)}_t\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\ ,$$

agent j's conditional expectation and variance of the capital gain on the i^{th} asset, respectively. As shown in the Appendix[2], under some not very restrictive simplifying assumptions the maximization problem of agent j, viz. $\max_{Z^{(j)}_{1,t}} E^{(j)}_t[\log(\Omega^{(j)}_{t+1})]$, results in the optimal investment fractions ($j\in\{f,c\}$)

$$Z^{(j)}_{1,t}=\frac{(m^{(j)}_{1,t}-m^{(j)}_{2,t})+(g_1-g_2)}{V^{(j)}_{1,t}}\ , \qquad Z^{(j)}_{2,t}=1-Z^{(j)}_{1,t}\ . \qquad (2)$$

The two groups of agents differ in the way they update their expectations over successive time intervals.

We assume that chartists follow the trend, so that their conditional expectations $m^{(c)}_{i,t}$, $i=1,2$, evolve over time according to the adaptive rule

$$m^{(c)}_{i,t+1}=(1-c_i)m^{(c)}_{i,t}+c_i\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right),$$

[1]The analysis can be generalized to the case where the utility functions are different for different agents with different risk coefficients, say, $u^{(j)}(\Omega)=(\Omega^{\lambda_j}-1)/\ \lambda_j$, $0<\lambda_j<1$.

[2]See also Chiarella and He (2001).

where $0 \leq c_i \leq 1$. On the other hand we assume that the fundamentalists, with their superior knowledge of the economy, are able to estimate the *fundamental* price of each asset at time t ($Y_{i,t}$, $i = 1, 2$) as the discounted present value of expected future payments. We also assume that they believe in a return to the fundamental price, so that their expectation of the capital gain on asset i over $(t, t+1)$ includes a time-varying short-run component $\eta_i(Y_{i,t} - P_{i,t})/P_{i,t}$, $0 \leq \eta_i \leq 1$, which is proportional to the (relative) deviation of the actual price $P_{i,t}$ from the fundamental $Y_{i,t}$, and a constant component ψ_i, that reflects fundamentalists' expectation about the growth rate of the fundamental in the long-run. Therefore the fundamentalist expected (relative) price change of the i-th asset may be written as

$$m_{i,t}^{(f)} = \eta_i(Y_{i,t}/P_{i,t} - 1) + \psi_i \ ,$$

so that:

$$m_{1,t}^{(f)} - m_{2,t}^{(f)} = \eta_1(Y_{1,t}/P_{1,t} - 1) - \eta_2(Y_{2,t}/P_{2,t} - 1) + \delta \ ,$$

where $\delta = \psi_1 - \psi_2$ represents the difference between the expected growth rates of the fundamentals.

3 Price setting rules

We assume the existence of a market maker, who is sufficiently "rational" to know the fundamental prices at time t as well as their growth rates over the next period (denote them by $\phi_{i,t+1}$, $i = 1, 2$). We assume that he/she also knows the wealth fractions that "in equilibrium" each type of agents will invest in each asset (denote them by $\widetilde{Z}_1^{(f)}, \widetilde{Z}_1^{(c)}, \widetilde{Z}_2^{(f)}, \widetilde{Z}_2^{(c)}$, with $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$)[3]. The market maker sets the excess demand of each asset to zero at the end of each trading period, by taking an offsetting long or short position, and adjusts the prices partly according to the out-of-equilibrium demand, and partly in order to steer the prices back to their fundamental values. The above assumptions about the market maker's behavior are expressed by the following price setting rule in the i-th market:

$$\begin{aligned} P_{i,t+1} - P_{i,t} \quad &= \quad \alpha_i(Y_{i,t} - P_{i,t}) + \phi_{i,t+1} Y_{i,t} + \qquad\qquad (3) \\ &\quad + P_{i,t}\beta_i \left(\frac{\Omega_t^{(f)}(Z_{i,t}^{(f)} - \widetilde{Z}_i^{(f)}) + \Omega_t^{(c)}(Z_{i,t}^{(c)} - \widetilde{Z}_i^{(c)})}{\Omega_t^{(f)} + \Omega_t^{(c)}} \right) , \end{aligned}$$

[3] We could for instance think of the market maker using his/her knowledge of the time series of the order flow from the two groups of market participants to estimate these fractions as a long-run average.

where $\beta_i > 0$, $0 < \alpha_i < 1$ and $i = 1, 2$. The price setting rule (3) can be interpreted in the sense that the market maker varies the price from period to period so as to adjust his/her inventory of the asset by raising the price when excess demand reduces the inventory and by lowering the price when excess supply determines inventory accumulation; at the same time, the market maker corrects his/her response so as to bring about "orderly" price movements in the market, this being one of the other assumed roles of the market maker[4]. In equation (3), the first term on the right hand side describes the market maker's policy that seeks to steer the price of the i-th asset back to fundamental price $Y_{i,t}$ (with adjustment coefficient α_i). The second term on the right hand side of (3) specifies the price change due to the underlying trend in the fundamental, assumed to be known to the market maker: this term represents the "equilibrium" portion of the price change. The third term on the right hand side is the portion of the price change depending on agents' demand at time t: consistent with our assumptions about the market maker's behaviour, this latter term is proportional to the out-of equilibrium demand for the i^{th} asset[5]. The coefficient β_i represents the market maker's speed of adjustment of the i^{th} price. Notice that in the absence of price adjustments due to the excess demand ($\beta_i \to 0$), the price setting rule (3) results in the difference equation:

$$P_{i,t+1} = P_{i,t} + \alpha_i(Y_{i,t} - P_{i,t}) + \phi_{i,t+1} Y_{i,t}$$

whose solution is

$$P_{i,t} = Y_{i,t} + (P_{i,0} - Y_{i,0})(1 - \alpha_i)^t$$

with $\lim_{t\to\infty} P_{i,t} = Y_{i,t}$, which means price converging from above or below to fundamental value according to the sign of the initial deviation $(P_{i,0} - Y_{i,0})$.

[4]In order to focus on the dynamics resulting from agents' interaction, the role played by the market maker in this model is highly stylised. The literature on the market maker behaviour suggests that he/she may play a more complex role in price formation, being not only a dealer who adjusts the quoted prices, but also an active investor (see, for instance, Madhavan and Smidt, 1993 and Madhavan, 2000).

[5]Since in this model agents' wealth, and therefore average demand for each asset, are growing over time, in eq. (3) we normalize the out-of-equilibrium demand, dividing it by total asset demand, i.e. by total agents' wealth $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$.

4 The dynamical system

The dynamics of the model derived in the previous sections can be summarized as

$$\begin{aligned}
P_{i,t+1} - P_{i,t} &= P_{i,t}\beta_i \left(\frac{\Omega_t^{(f)}(Z_{i,t}^{(f)} - \widetilde{Z}_i^{(f)}) + \Omega_t^{(c)}(Z_{i,t}^{(c)} - \widetilde{Z}_i^{(c)})}{\Omega_t^{(f)} + \Omega_t^{(c)}} \right) + \\
&\quad + \alpha_i (Y_{i,t} - P_{i,t}) + \phi_{i,t+1} Y_{i,t}, \qquad (i = 1, 2) \\
m_{i,t+1}^{(c)} &= (1 - c_i) m_{i,t}^{(c)} + c_i \left(\frac{P_{i,t+1} - P_{i,t}}{P_{i,t}} \right), \qquad (i = 1, 2), \\
\Omega_{t+1}^{(j)} &= \Omega_t^{(j)} + \Omega_t^{(j)} \sum_{i=1}^{2} Z_{i,t}^{(j)} \left(\frac{P_{i,t+1} - P_{i,t}}{P_{i,t}} + g_i \right), \quad j \in \{f, c\} \qquad (4)
\end{aligned}$$

where

$$Z_{1,t}^{(f)} = \frac{\eta_1 (Y_{1,t}/P_{1,t} - 1) + \delta - \eta_2 (Y_{2,t}/P_{2,t} - 1) + (g_1 - g_2)}{V_{1,t}^{(f)}}, \qquad (5)$$

$$Z_{1,t}^{(c)} = \frac{(m_{1,t}^{(c)} - m_{2,t}^{(c)}) + (g_1 - g_2)}{V_{1,t}^{(c)}}, \qquad (6)$$

and $Z_{2,t}^{(j)} = 1 - Z_{1,t}^{(j)}$, $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$.

For the moment we do not specify the way in which agents update their beliefs about the conditional variance in eqs. (5) and (6). In order to avoid adding more dynamic equations to the model, we assume that agent type j calculates $V_{1,t}^{(j)}$ as a function of the state of the system at time t, in particular we assume that $V_{1,t}^{(j)}$ is a function of the expected returns[6] $m_{1,t}^{(j)}$ and $m_{2,t}^{(j)}$. On the other hand, the specification of such a function is only needed in order to perform numerical simulations, but it is not required in order to transform the model into a stationary system and to analyse its general properties (see the following subsections).

[6]This general assumption includes the case where the conditional variance $V_{1,t}^{(j)}$ is constant, or the more realistic case where it varies over time together with the size of the expected absolute excess return $\left|(m_{1,t}^{(j)} - m_{2,t}^{(j)}) + (g_1 - g_2)\right|$.

4.1 The stationary model

In this model both prices and wealth processes are growing, due to the underlying growth of the fundamentals in each market and due to the fact that the optimal demand for each asset depends on agents' wealth. A stationary system can be obtained by formulating the model in terms of *returns* and *wealth shares*. To do this, we consider the changes of variables[7]

$$\begin{aligned} w_t^{(f)} &= \frac{\Omega_t^{(f)}}{\Omega_t^{(f)} + \Omega_t^{(c)}} \ , \quad w_t^{(c)} = \frac{\Omega_t^{(c)}}{\Omega_t^{(f)} + \Omega_t^{(c)}} = 1 - w_t^{(f)} \ , \\ \rho_{i,t+1} &= \frac{P_{i,t+1} - P_{i,t}}{P_{i,t}}, \quad y_{i,t} = Y_{i,t}/P_{i,t} \ , \quad i = 1,2 \ , \end{aligned}$$

where $w_t^{(f)}$ and $w_t^{(c)}$ are the wealth proportions of fundamentalists and chartists at time t, respectively, $y_{i,t}$ is the ratio between the fundamental value and the price of the i-th asset, $\rho_{i,t+1}$ is the capital gain on the i-th asset over $(t, t+1)$, and therefore $(\rho_{i,t+1} + g_i)$ represents the return on the i-th asset, $i = 1, 2$. We also denote by $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$ the total wealth at time t, and by $\omega_{t+1}^{(j)}$, $j \in \{f, c\}$, the growth rate of the wealth of agent type j over $(t, t+1)$, so that

$$\omega_{t+1}^{(j)} = \frac{\Omega_{t+1}^{(j)} - \Omega_t^{(j)}}{\Omega_t^{(j)}} = (1 - Z_{1,t}^{(j)})(\rho_{2,t+1} + g_2) + Z_{1,t}^{(j)}(\rho_{1,t+1} + g_1) \ . \qquad (7)$$

Using the foregoing changes of variables, equations (4) may be rewritten in terms of total wealth, wealth proportions of the two groups, and rates of growth of the agents' wealth, giving

$$w_{t+1}^{(f)} = w_t^{(f)}(1 + \omega_{t+1}^{(f)})\frac{\Omega_t}{\Omega_{t+1}} \ , \quad w_{t+1}^{(c)} = w_t^{(c)}(1 + \omega_{t+1}^{(c)})\frac{\Omega_t}{\Omega_{t+1}} \ . \qquad (8)$$

Summing up the last two equations, and recalling that $w_t^{(c)} = 1 - w_t^{(f)}$, we obtain

[7]Similar changes of variables for prices and wealth are used, in order to get a stationary model of asset price dynamics, by Chiarella and He (2001). An evolutionary model of a financial market formulated in terms of *wealth shares* of the market participants is also developed, in a different framework, by Blume and Easley (1992).

$$\Omega_{t+1} = w_t^{(f)}\Omega_t(1+\omega_{t+1}^{(f)}) + (1-w_t^{(f)})\Omega_t(1+\omega_{t+1}^{(c)}) \tag{9}$$

Substituting into the first equation of (8), the law governing the time evolution of the wealth proportion of fundamentalists (and chartists) is obtained, namely

$$\begin{aligned} w_{t+1}^{(f)} &= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{w_t^{(f)}(1+\omega_{t+1}^{(f)}) + (1-w_t^{(f)})(1+\omega_{t+1}^{(c)})} \\ &= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{1+\omega_{t+1}^{(c)} + w_t^{(f)}(\omega_{t+1}^{(f)} - \omega_{t+1}^{(c)})}, \end{aligned} \tag{10}$$

and $w_{t+1}^{(c)} = (1 - w_{t+1}^{(f)})$.

As far as the time evolution of the fundamental prices $Y_{i,t}$ is concerned, we assume that their growth rates (assumed to be known to the market maker) are constant over time, $\phi_{i,t+1} = \phi_i$, $i = 1, 2$. It follows that the fundamental/price ratios evolve according to

$$y_{i,t+1} = \frac{(1+\phi_i)}{(1+\rho_{i,t+1})} y_{i,t}, \qquad i = 1, 2\ .$$

Recalling also that $Z_{2,t}^{(j)} = 1 - Z_{1,t}^{(j)}$ and $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$, we obtain the stationary dynamical system

$$\begin{aligned} \rho_{1,t+1} &= \beta_1[w_t^{(f)}(Z_{1,t}^{(f)} - \widetilde{Z}_1^{(f)}) + (1-w_t^{(f)})(Z_{1,t}^{(c)} - \widetilde{Z}_1^{(c)})] + \\ &\quad +\alpha_1(y_{1,t}-1) + \phi_1 y_{1,t}, \\ \rho_{2,t+1} &= \beta_2[w_t^{(f)}(\widetilde{Z}_1^{(f)} - Z_{1,t}^{(f)}) + (1-w_t^{(f)})(\widetilde{Z}_1^{(c)} - Z_{1,t}^{(c)})] + \\ &\quad +\alpha_2(y_{2,t}-1) + \phi_2 y_{2,t}, \\ y_{1,t+1} &= \frac{(1+\phi_1)}{(1+\rho_{1,t+1})} y_{1,t}, \\ y_{2,t+1} &= \frac{(1+\phi_2)}{(1+\rho_{2,t+1})} y_{2,t}, \\ m_{1,t+1}^{(c)} &= (1-c_1)m_{1,t}^{(c)} + c_1\rho_{1,t+1}, \\ m_{2,t+1}^{(c)} &= (1-c_2)m_{2,t}^{(c)} + c_2\rho_{2,t+1}, \\ w_{t+1}^{(f)} &= \frac{w_t^{(f)}(1+\omega_{t+1}^{(f)})}{w_t^{(f)}(1+\omega_{t+1}^{(f)}) + (1-w_t^{(f)})(1+\omega_{t+1}^{(c)})}, \end{aligned}$$

where $\omega_{t+1}^{(j)}$, $j \in \{f, c\}$, $Z_{1,t}^{(f)}$ and $Z_{1,t}^{(c)}$ are given by (7), (5) and (6), respectively.

From our assumptions about the market maker's price setting rules it follows that when prices and agents' investment fractions are at their equilibrium levels ($y_{i,t} = 1$, $i = 1, 2$, $Z_{1,t}^{(j)} = \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$), then the actual returns and the chartists' expected returns are equal to the rates of growth of the fundamentals; i.e. $\rho_{i,t} = m_{i,t}^{(c)} = \phi_i$, $i = 1, 2$. As a consequence, the equilibrium optimal investment fractions are given by

$$\widetilde{Z}_1^{(f)} = \frac{\delta + (g_1 - g_2)}{\widetilde{V}_1^{(f)}} , \tag{12}$$

$$\widetilde{Z}_1^{(c)} = \frac{(\phi_1 - \phi_2) + (g_1 - g_2)}{\widetilde{V}_1^{(c)}} . \tag{13}$$

and $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$ ($\widetilde{V}_1^{(j)}$represents the belief of investor type j about the conditional variance of the risky return in equilibrium). Eqs. (12) and (13) state that the equilibrium investment fraction in the risky asset is given by the expected equilibrium risk-adjusted excess return, as one would expect.

4.2 The map

As we are assuming that agent j calculates $V_{1,t}^{(j)}$ as a function of the expected returns $m_{1,t}^{(j)}$ and $m_{2,t}^{(j)}$, it follows that we can represent the fundamentalist conditional variance and the fundamentalist optimal investment fractions as functions of the state variables y_1 and y_2 according to

$$\begin{aligned} V_{1,t}^{(f)} &= v^{(f)}(y_{1,t}, y_{2,t}) , \\ Z_{1,t}^{(f)} &= \zeta^{(f)}(y_{1,t}, y_{2,t}) = \frac{\eta_1(y_{1,t} - 1) + \delta - \eta_2(y_{2,t} - 1) + (g_1 - g_2)}{v^{(f)}(y_{1,t}, y_{2,t})} . \end{aligned}$$

Analogously, we can represent the chartist conditional variance and the chartist optimal investment fractions as functions of the state variables $m_1^{(c)}$ and $m_2^{(c)}$ by

$$\begin{aligned} V_{1,t}^{(c)} &= v^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)}) , \\ Z_{1,t}^{(c)} &= \zeta^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)}) = \frac{(m_{1,t}^{(c)} - m_{2,t}^{(c)}) + (g_1 - g_2)}{v^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)})} . \end{aligned}$$

In this notation the equilibrium fractions (12) and (13) would be given by $\widetilde{Z}_1^{(f)} = \zeta^{(f)}(1,1)$ and $\widetilde{Z}_1^{(c)} = \zeta^{(c)}(\phi_1, \phi_2)$, respectively.

To summarize, by denoting with the symbol $'$ the unit time advancement operator[8], the time evolution of the model is given by the iteration of the following 7-dimensional nonlinear map T

$T\ :\ (\rho_1, \rho_2, y_1, y_2, m_1^{(c)}, m_2^{(c)}, w^{(f)}) \longmapsto (\rho'_1, \rho'_2, y'_1, y'_1, m_1^{(c)\prime}, m_2^{(c)\prime}, w^{(f)\prime})$: defined by

$$T: \begin{cases} \rho'_1 = \beta_1[w^{(f)}(Z_1^{(f)} - \widetilde{Z}_1^{(f)}) + (1 - w^{(f)})(Z_1^{(c)} - \widetilde{Z}_1^{(c)})] + \alpha_1(y_1 - 1) + \phi_1 y_1, \\ \rho'_2 = \beta_2[w^{(f)}(\widetilde{Z}_1^{(f)} - Z_1^{(f)}) + (1 - w^{(f)})(\widetilde{Z}_1^{(c)} - Z_1^{(c)})] + \alpha_2(y_2 - 1) + \phi_2 y_2, \\ y'_1 = \frac{(1+\phi_1)}{(1+\rho'_1)} y_1, \\ y'_2 = \frac{(1+\phi_2)}{(1+\rho'_2)} y_2, \\ m_1^{(c)\prime} = (1 - c_1) m_1^{(c)} + c_1 \rho'_1, \\ m_2^{(c)\prime} = (1 - c_2) m_2^{(c)} + c_2 \rho'_2, \\ w^{(f)\prime} = \frac{w^{(f)}[1+(1-Z_1^{(f)})(\rho'_2+g_2)+Z_1^{(f)}(\rho'_1+g_1)]}{1+(1-Z_1^{(c)})(\rho'_2+g_2)+Z_1^{(c)}(\rho'_1+g_1)+w^{(f)}(Z_1^{(f)}-Z_1^{(c)})(\rho'_1+g_1-\rho'_2-g_2)}, \end{cases} \tag{14}$$

where

$$Z_1^{(f)} \equiv \zeta^{(f)}(y_1, y_2) = \frac{\eta_1(y_1 - 1) + \delta - \eta_2(y_2 - 1) + (g_1 - g_2)}{v^{(f)}(y_1, y_2)} \ ,$$

$$Z_1^{(c)} \equiv \zeta^{(c)}(m_1^{(c)}, m_2^{(c)}) = \frac{m_1^{(c)} - m_2^{(c)} + (g_1 - g_2)}{v^{(c)}(m_1^{(c)}, m_2^{(c)})} \ ,$$

with $\widetilde{Z}_1^{(f)} = \zeta^{(f)}(1,1)$, $\widetilde{Z}_1^{(c)} = \zeta^{(c)}(\phi_1, \phi_2)$[9].

5 Invariant subsets of the phase space and fundamental equilibria

In this section we focus on some general properties of the map (14), that can help us to understand the dynamic behaviour of the system. An important

[8] i.e. if x is the value of a state variable at time t, then x' denotes the value of the same variable at time $(t + 1)$.

[9] We note that in the first two equations of (14) the variables ρ'_1 and ρ'_2 do not depend on ρ_1 and ρ_2, but only on y_1, y_2, $m_1^{(c)}$, $m_2^{(c)}$, $w^{(f)}$, so the dynamic variables ρ_1 and ρ_2 could be eliminated by substitution, thus giving a 5-dimensional system. However in this case we would obtain very complicated analytical expressions for the remaining dynamic equations, and since the state variables ρ_1 and ρ_2 are the ones on which we focus our attention, we analyse the dynamic model in the form (14).

feature of the map is the existence of a one-dimensional invariant subset of the phase space, associated with the "fundamental" levels of the state variables ρ_i, $m_i^{(c)}$, y_i, $i = 1, 2$. In fact, assume that such variables are at their equilibrium levels, i.e. $\rho_1 = m_1^{(c)} = \phi_1$, $\rho_2 = m_2^{(c)} = \phi_2$, $y_1 = y_2 = 1$. Then it is easy to check that such variables do not vary under iteration by the map T, i.e.

$$T(\phi_1, \phi_2, 1, 1, \phi_1, \phi_2, w^{(f)}) = (\phi_1, \phi_2, 1, 1, \phi_1, \phi_2, w^{(f)\prime}) \ ,$$

which means that the dynamics of the system are constrained in a one-dimensional subset of the phase space (let us denote it by E). The subset E of the phase space is *invariant*, and along such an invariant manifold the time evolution of the system is obtained by iteration of a one-dimensional nonlinear map governing the dynamics of the wealth fractions, say $T^{(w)}$: $w^{(f)} \longmapsto w^{(f)\prime}$, given by

$$w^{(f)\prime} = \frac{w^{(f)}(1 + \widetilde{\omega}^{(f)})}{1 + \widetilde{\omega}^{(c)} + w^{(f)}(\widetilde{\omega}^{(f)} - \widetilde{\omega}^{(c)})} \ , \tag{15}$$

where

$$\begin{aligned} \widetilde{\omega}^{(f)} &= (1 - \widetilde{Z}_1^{(f)})(\phi_2 + g_2) + \widetilde{Z}_1^{(f)}(\phi_1 + g_1) \ , \\ \widetilde{\omega}^{(c)} &= (1 - \widetilde{Z}_1^{(c)})(\phi_2 + g_2) + \widetilde{Z}_1^{(c)}(\phi_1 + g_1), \end{aligned}$$

are the rates of growth of the wealth of fundamentalists and chartists along the invariant manifold. Notice that, apart from the particular case where $\widetilde{Z}_1^{(f)} = \widetilde{Z}_1^{(c)}$, and thus $\widetilde{\omega}^{(f)} = \widetilde{\omega}^{(c)}$, the one-dimensional map (15) admits two fixed points, $w^{(f)} = 1$ and $w^{(f)} = 0$[10]. We can conclude that the system has at least two steady states (let us denote them by *fundamental* steady states, F and C), both lying in the invariant subset E, that are characterized by "fundamental" equilibrium levels of the state variables

$$\rho_1 = m_1^{(c)} = \phi_1 \ , \quad \rho_2 = m_2^{(c)} = \phi_2 \ , \quad y_1 = y_2 = 1 \ ,$$

[10] In the particular case where $\widetilde{Z}_1^{(f)} = \widetilde{Z}_1^{(c)}$ any value $\overline{w}^{(f)}$, $0 \leq \overline{w}^{(f)} \leq 1$ is an equilibrium value for the state variable $w^{(f)}$. In the general case where $\widetilde{Z}_1^{(f)} \neq \widetilde{Z}_1^{(c)}$, the convergence of the system to either one or the other of the two equilibria depends on the relation between $\widetilde{\omega}^{(f)}$ and $\widetilde{\omega}^{(c)}$, i.e. between the equilibrium investment fractions $\widetilde{Z}_1^{(f)}$ and $\widetilde{Z}_1^{(c)}$. Assume, in particular, $\phi_1 + g_1 > \phi_2 + g_2$: in the case $\widetilde{Z}_1^{(f)} > \widetilde{Z}_1^{(c)}$, i.e. $\widetilde{\omega}^{(f)} > \widetilde{\omega}^{(c)}$(fundamentalists are investing more than chartists in the asset with higher return in equilibrium), $w^{(f)} = 0$ is unstable and $w^{(f)} = 1$ is a (globally) stable equilibrium (chartists disappear in the long run), while in the opposite case, $\widetilde{Z}_1^{(f)} < \widetilde{Z}_1^{(c)}$, i.e. $\widetilde{\omega}^{(f)} < \widetilde{\omega}^{(c)}$, $w^{(f)} = 1$ is unstable and $w^{(f)} = 0$ is a (globally) attracting equilibrium (fundamentalists disappear in the long-run).

and by equilibrium wealth proportions $w^{(f)} = 1$ or $w^{(f)} = 0$, where only fundamentalists or, respectively, chartists survive in the market. Of course such situations cannot be considered as true equilibrium situations; they are a result of the fact that our dynamic model does not consider the possible time evolution of the size of the two groups according, e.g., to the realized profits. On the other hand, numerical simulations show that if optimal agents' investment fractions in equilibrium $\widetilde{Z}_1^{(f)}$ and $\widetilde{Z}_1^{(c)}$ are not too different from each other, then the wealth dynamics along the invariant manifold E is very slow. It seems more important to analyse the conditions (values of the parameters and initial state of the system) under which, either the system converges to the the invariant subset E (with prices and returns settling down on their "fundamental" levels), or it converges to other attracting sets characterized by fluctuations in prices, returns and wealth fractions. This is what we will do in the next section through numerical simulations.

Other invariant subsets of the phase space are those associated with the cases $w^{(f)} = 1$ (where only fundamentalists operate in the market) and $w^{(f)} = 0$ (only chartists operate in the market). Notice that in both cases the dynamics of the system are obtained by iteration of a 6-dimensional map. In particular in the first case, $w^{(f)} = 1$, the dynamic equations for $m_1^{(c)}$ and $m_2^{(c)}$ are not meaningful (and do not feed back into the system) and thus the dynamics are obtained by iteration of a 4-dimensional map.

Particularly interesting is the case $w^{(f)} = 0$. The 6-dimensional map, say $T^{(c)}$ driving the system in this case is given by

$$T^{(c)} : \begin{cases} \rho_1' = \beta_1(Z_1^{(c)} - \widetilde{Z}_1^{(c)}) + \alpha_1(y_1 - 1) + \phi_1 y_1, \\ \rho_2' = \beta_2(\widetilde{Z}_1^{(c)} - Z_1^{(c)}) + \alpha_2(y_2 - 1) + \phi_2 y_2, \\ y_1' = \frac{(1+\phi_1)}{(1+\rho_1')} y_1, \\ y_2' = \frac{(1+\phi_2)}{(1+\rho_2')} y_2, \\ m_1^{(c)\prime} = (1 - c_1) m_1^{(c)} + c_1 \rho_1', \\ m_2^{(c)\prime} = (1 - c_2) m_2^{(c)} + c_2 \rho_2'. \end{cases} \tag{16}$$

Numerical simulations show that the fundamental steady state $\rho_1 = m_1^{(c)} = \phi_1$, $\rho_2 = m_2^{(c)} = \phi_2$, $y_1 = y_2 = 1$, is locally asymptotically stable for sufficiently low values of the chartists adjustment parameters c_1 and c_2, and of the speeds of adjustment of market prices β_1 and β_2, while for higher values of these adjustment parameters the equilibrium becomes unstable and trajectories converge to an attracting limit cycle, with persistent fluctuations around the fundamental equilibrium. The results of these numerical experiments are given in the next section, where it is also shown how the study of the dynamics of the system in the limiting case $w^{(f)} = 0$ is a good starting point to understand the dynamic behaviour of the full system.

6 Out-of-equilibrium dynamics and coexistence of attracting sets

Throughout the present section, we analyse numerically some global dynamic phenomena of the system; these include "non fundamental" equilibria, oscillatory dynamics (regular of chaotic), and coexistence of attracting sets in the phase space. In order to perform the numerical simulations, we specify the analytical form of the fundamentalist and chartist investment fractions $Z_1^{(f)} = \zeta^{(f)}(y_1, y_2)$ and $Z_1^{(c)} = \zeta^{(c)}(m_1^{(c)}, m_2^{(c)})$. Similarly to the basic model developed in Chiarella, Dieci and Gardini (2002), we assume that the fundamentalist estimate of the conditional variance of the risky return is constant ($V_1^{(f)} = \overline{v}_1^{(f)}$), so that the fundamentalist optimal investment proportion in asset 1 is a linear function of the deviations from fundamentals:-

$$\zeta^{(f)}(y_1, y_2) = a_1(y_1 - 1) - a_2(y_2 - 1) + b \ ,$$

where $a_i = \eta_i/\overline{v}_1^{(f)}$, $i = 1, 2$, and $b = (\delta + g_1 - g_2)/\overline{v}_1^{(f)}$. Unlike the fundamentalists, the chartists are assumed to change their estimate $V_1^{(c)}$ of the conditional variance according to the magnitude of the expected excess return $\left|m_1^{(c)} - m_2^{(c)}\right|$. As this quantity becomes larger they expect greater volatility and increase their estimate $V_1^{(c)}$, so that the optimal investment proportion in asset 1 results in a nonlinear S-shaped increasing function of the expected excess return. For the numerical experiments we use

$$\zeta^{(c)}(m_1^{(c)}, m_2^{(c)}) = h(m_1^{(c)} - m_2^{(c)} + g_1 - g_2) \ ,$$

where

$$h(x) = \frac{\gamma}{\theta} \arctan(\theta x) \ , \qquad (\gamma, \theta > 0) \ .$$

Fig. 1 displays the shape of the function $h(x)$ with different values of the parameters γ, θ[11].

[11] The parameter $\gamma = h'(0)$ governs the slope of the chartist demand function (strength of chartist demand) for $x = 0$, and the coefficient $1/\gamma$ can be interpreted as the estimate of the variance in the case of zero expected excess return. The parameters γ and θ jointly determine the "floor" and the "ceiling" of the chartist investment fractions.

Fig. 1

parameters

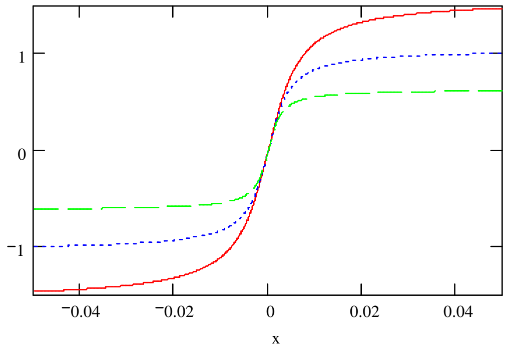
$\gamma_0 = 200$
$\theta_1 = 200$
$\theta_2 = 300$
$\theta_3 = 500$

chartist optimal investment fraction in asset 1 for different values of the parameter θ

$h_1(x) = \frac{\gamma_0}{\theta_1} \cdot \mathrm{atan}(\theta_1 \cdot x)$

$h_2(x) = \frac{\gamma_0}{\theta_2} \cdot \mathrm{atan}(\theta_2 \cdot x)$

$h_3(x) = \frac{\gamma_0}{\theta_3} \cdot \mathrm{atan}(\theta_3 \cdot x)$



parameters

$\theta_0 = 250$
$\gamma_1 = 100$
$\gamma_2 = 200$
$\gamma_3 = 300$

chartist optimal investment fraction in asset 1 for different values of the parameter γ

$h_1(x) = \frac{\gamma_1}{\theta_0} \cdot \mathrm{atan}(\theta_0 \cdot x)$

$h_2(x) = \frac{\gamma_2}{\theta_0} \cdot \mathrm{atan}(\theta_0 \cdot x)$

$h_3(x) = \frac{\gamma_3}{\theta_0} \cdot \mathrm{atan}(\theta_0 \cdot x)$

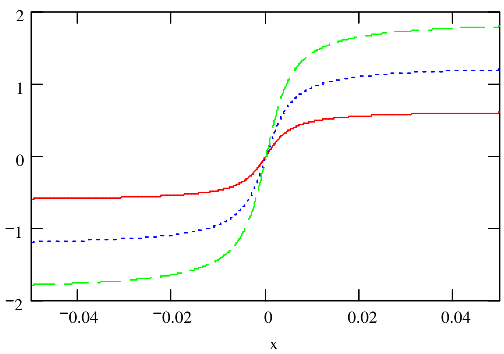


Figure 1: chartist demand function.

6.1 Non fundamental equilibria

Depending on the parameters that characterize the agents' demand functions, the map T may admit further fixed points (besides the *fundamental* steady states F and C), characterized by price/fundamental ratios tending to infinity, i.e. by $y_1 = 0$ or $y_2 = 0$. As an example, with the demand functions assumed above, it is easy to check that the point $y_1 = 0$, $\rho_2 = m_2^{(c)} = \phi_2$, $w^{(f)} = 0$, $\rho_1 = m_1^{(c)} = \overline{\rho}_1$, $y_2 = \overline{y}_2$, where $\overline{\rho}_1$ solves

$$\overline{\rho}_1 + \alpha_1 = \beta_1 \frac{\gamma}{\theta} \left\{ \arctan[\theta(\overline{\rho}_1 - \phi_2 + g_1 - g_2)] - \arctan[\theta(\phi_1 - \phi_2 + g_1 - g_2)] \right\}$$

and

$$\overline{y}_2 = 1 + \frac{\beta_2(\overline{\rho}_1 + \alpha_1)}{\beta_1(\phi_2 + \alpha_2)},$$

is an equilibrium point for the dynamical system. *Fig. 2* represents the projection, in the plane of the state variables ρ_1, y_1, of a trajectory converging to such a *non fundamental* solution, where the price of the risky asset grows at the rate $\rho_1 = \overline{\rho}_1 \simeq 0.21079$ (higher than the rate of growth of the fundamental value $\phi_1 = 0.02$), and the fundamental/price ratios are $y_1 = 0$, $y_2 = \overline{y}_2 \simeq 1.03389$. In such an example, although the initial values of the state variables are taken very close to the fundamental levels, and agents adjust their expectations weakly, the price of the risky asset increasingly deviates from the fundamental value ($y_1 \to 0$) due to the high speed of reaction of the price to the excess demand (high value of β_1). Further numerical simulations show that the possibility that the system converges to such equilibria is ruled out provided that the reaction parameters of market prices to excess demand (β_i, $i = 1, 2$) are not too high.

Fig. 2

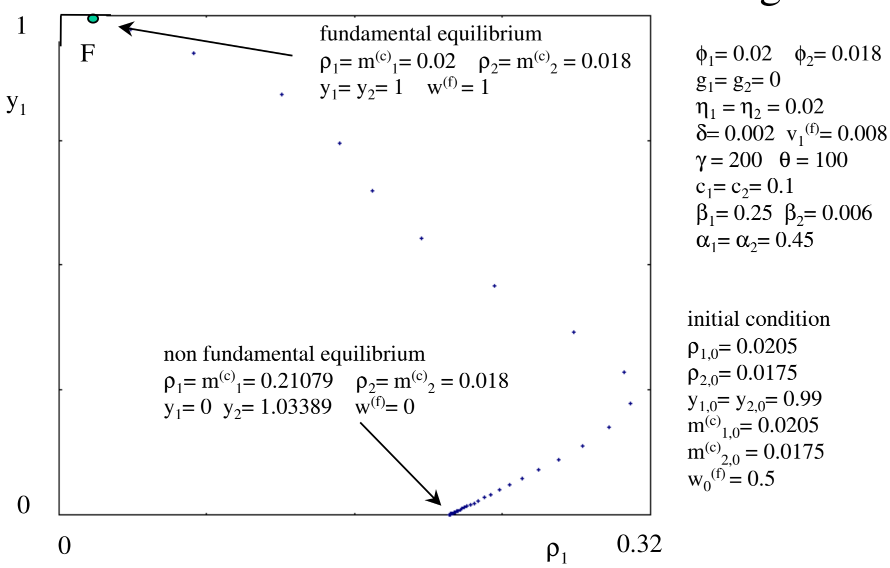


Figure 2: convergence to a non fundamental equilibrium.

6.2 Steady state bifurcations and attracting limit cycles

For particular parameter values, the long-run evolution of the system may be characterized by stable oscillations along a limit cycle, where prices and returns fluctuate around their fundamental levels. Let us first illustrate such phenomenon in the subcase $w^{(f)} = 0$, where only chartists operate in the market and therefore, as stressed in the previous section, the time evolution of the system is obtained by iteration of the lower-dimensional map $T^{(c)}$ given by (16). *Fig. 3a* shows the projection, in the plane of the state variables ρ_1, y_1, of a trajectory converging with damped oscillations to the "fundamental equilibrium", which is an attracting focus[12]. By increasing the values of the adjustment parameters c_1, c_2, β_1 and β_2, with respect to the ones used in *Fig. 3a*, the fundamental equilibrium becomes a repelling

[12]The fact that the system may converge to the fundamental steady state, although only chartists survive in the market, is due to the assumed stabilizing role of the market maker.

focus and trajectories converge to an attracting limit cycle, existing around the equilibrium. We have found numerical evidence that the creation of such a limit cycle occurs through a supercritical Neimark-Hopf bifurcation. Figs. *3b* and *3c* show the attracting invariant closed curve existing for increased values of the parameters c_1 and, respectively, β_1 starting from the stable case represented in Fig.*3a*.

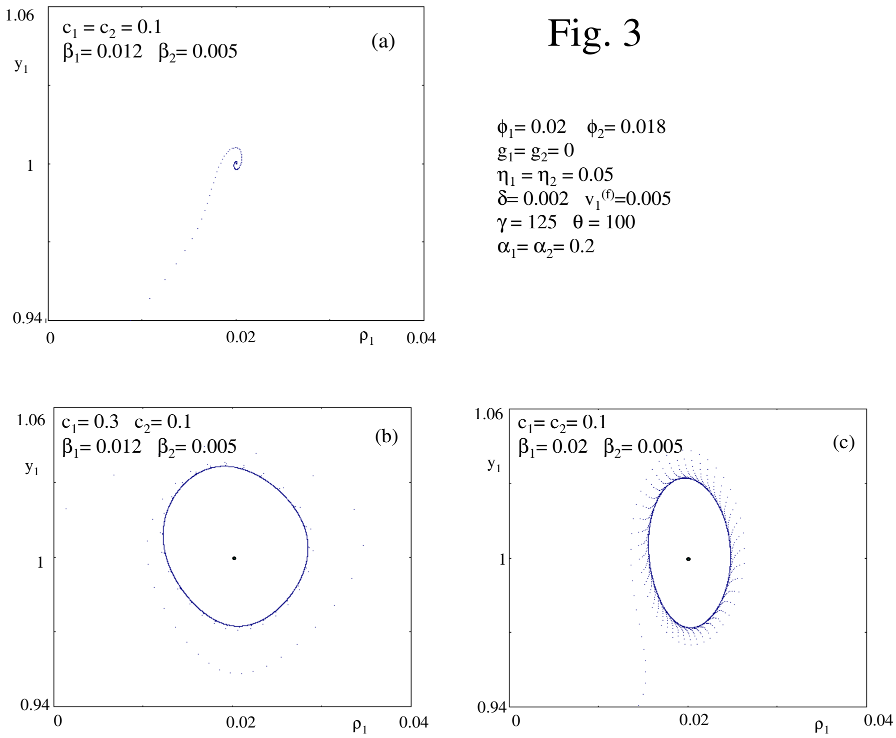


Figure 3: market dominated by chartists;

(a) *low values of the adjustment parameters: convergence to the "fundamental" equilibrium;*

(b) *higher value of the chartist adjustment parameter;*

(c) *higher speed of adjustment of market price: convergence to a limit cycle around the fundamental equilibrium.*

As outlined in the previous section, the lower dimensional system obtained in the case $w^{(f)} = 0$ is a good starting point from which to understand the dynamic behaviour of the full system. In fact, the attracting limit cycle of the lower dimensional map (16) may also be reached by trajectories starting with positive initial fundamentalist wealth fraction $w_0^{(f)}$. In *Fig. 4* we take parameter values similar to the ones used in *Fig. 3a*. *Fig. 4a* shows a case with high initial proportion of chartist wealth, $w_0^{(f)} = 0.1$; the trajectory converges to the attracting limit cycle in the 6-dimensional invariant subset $w^{(f)} = 0$. On the contrary, *Fig. 4b* shows a trajectory with high initial proportion of fundamentalist wealth, $w_0^{(f)} = 0.9$, converging to the "fundamental" invariant set E (with fundamentalist wealth fraction converging to $w^{(f)} = 1$). This is a phenomenon of coexistence of an attracting equilibrium and an attracting limit cycle. Of course, in the case of coexistence of attracting sets the numerical study of the basins of attraction becomes crucial, with particular attention to the effect, on the dynamic outcome of the system, of the initial wealth proportion of the two groups and of the initial deviation of the prices from the fundamentals.

Other *global* dynamic phenomena, in particular new cases of coexistence of attracting sets, will be analysed in the next section, by use of numerical simulations.

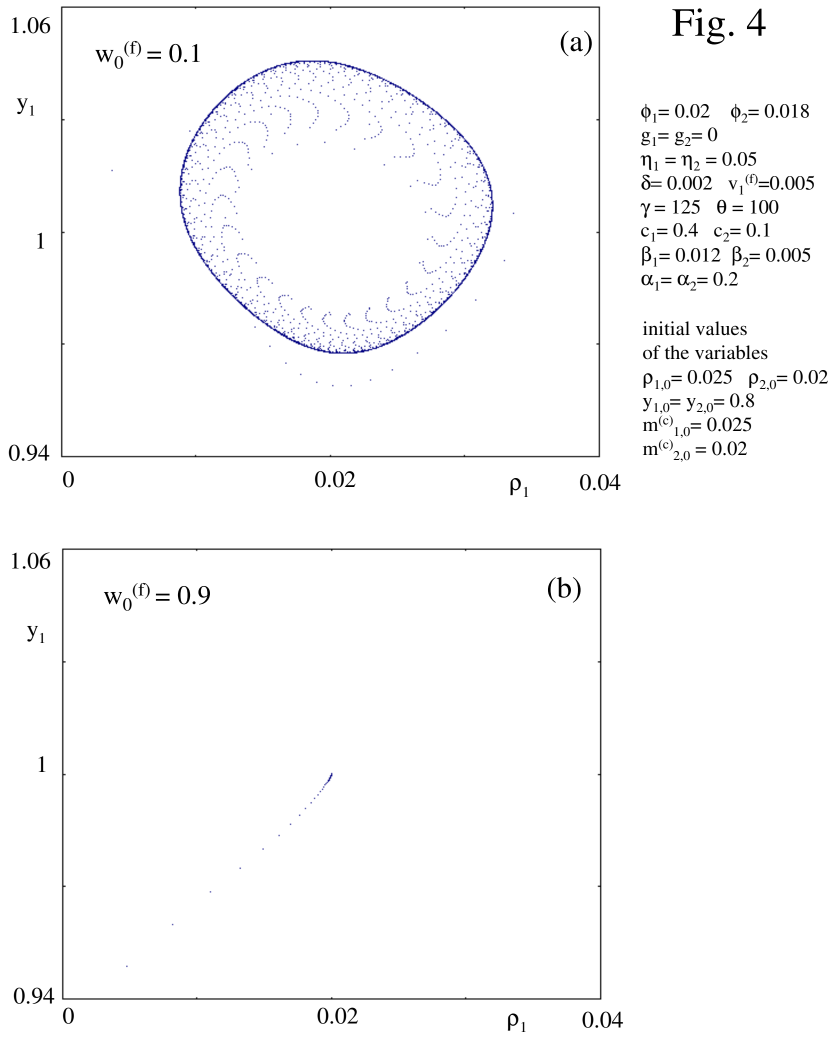


Figure 4:

(a) *high initial proportion of chartists' wealth: convergence to a limit cycle around the fundamental equilibrium;*

(b) *high initial proportion of fundamentalists' wealth: convergence to the fundamental equilibrium.*

6.3 Global dynamics and bifurcations

In this section we focus on some important *global* phenomena, such as the creation of new attractors, characterized by regular or chaotic oscillations, and the effect on their structure of changes of the key parameters of the model. Although the numerical examples of this section are obtained with particular values of the parameters, the phenomena that we describe are persistent and can be easily observed in several parameter regimes.

The first phenomenon we consider concerns the existence of a new attracting limit cycle (different from the one that may exist in the invariant subset $w^{(f)} = 0$), characterized by *long-run fluctuations of the wealth proportions* of fundamentalists and chartists, as well as of the other remaining variables. The creation of such an attracting closed curve is in general due to a *global* bifurcation, i.e. it is not related to the eigenvalues of the Jacobian matrix of T evaluated at the equilibria F and C. An example of an attracting limit cycle of this kind is represented in *Fig. 5a.* Along the limit cycle both groups of agents survive in the long-run, with wealth proportions fluctuating around average values. Let us now analyse the effect of the key parameters characterizing the chartist behaviour, namely the chartist demand parameter γ and the adjustment parameters c_1 and c_2. *Figs. 5b* and *5c* represent the effect on the limit cycle of increasing the parameter γ, that governs the slope of the chartist demand function. Looking at the projection of the attractor in the (ρ_1, y_1) plane, we observe a transition from regular to chaotic fluctuations and an increasing average wealth proportion of fundamentalists as long as the fluctuations become chaotic. The latter effect is probably due to the fact that the adaptive rule used by chartists to forecast asset returns is more succesful in the case of regular, rather than erratic, fluctuations. A similar effect on the nature of the cyclic attractor and on the average wealth proportions in the long-run is obtained if we increase the adjustment parameters c_1 and c_2 as shown in *Figs. 5d* and *5e.*

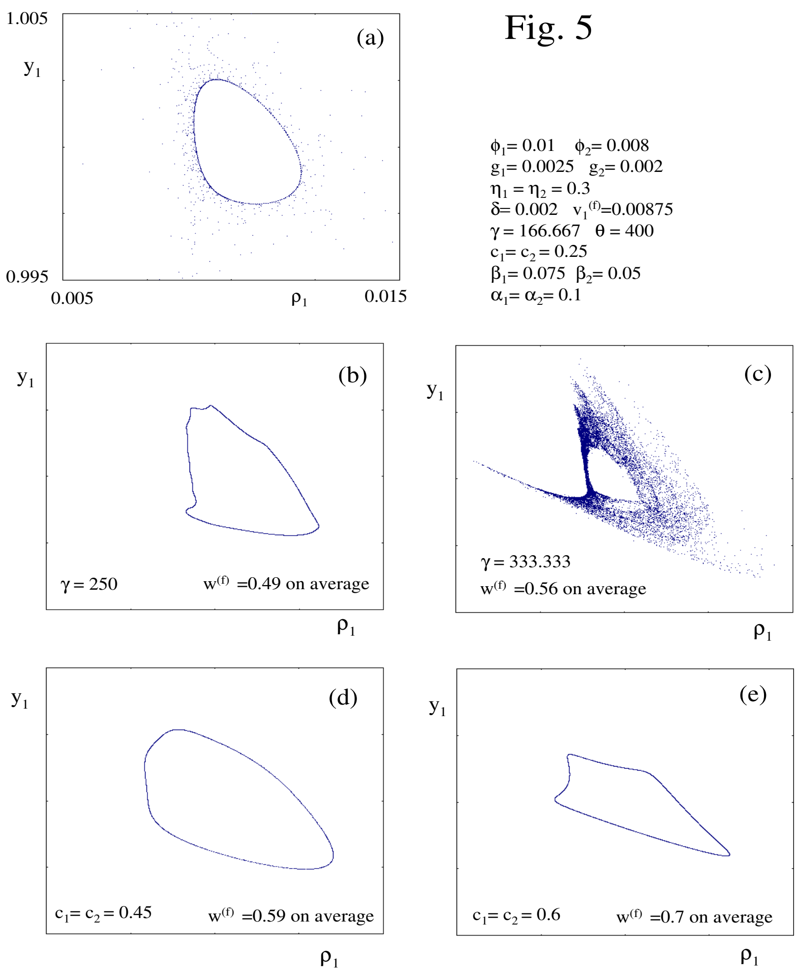


Figure 5:

(a) *prices, returns and wealth shares fluctuate on a limit cycle;*

(b),(c) *effect of increasing the strength of chartists demand* γ*;*

(d),(e) *effect of increasing the chartist adjustment parameters* c_1 *and* c_2.

Of course such an attractor may coexist with other possible asymptotic behaviours of the system. As an example, with the slightly different parameter set of *Fig. 6* an attracting limit cycle characterized by long-run fluctuations of prices, returns and wealth proportions, coexists in the phase space with the attracting "fundamental" invariant set E, where prices and returns settle down at their "fundamental" levels. The initial condition is the same in *Fig. 6a* and in *Fig. 6b*, except for the initial wealth proportion of fundamentalists ($w_0^{(f)} = 0.375$ in *Fig. 6a*, $w_0^{(f)} = 0.3$ in *Fig. 6b*). The system converges to the limit cycle only for a sufficiently low initial wealth proportion of the fundamentalists.

A second dynamic phenomenon that is worth considering is the existence of asymptotic behaviors where the system may switch between different regimes characterized by fluctuations of different amplitude. Such phenomenon may consist in a regular switching between phases of periodic fluctuations of different size, as in the case represented in *Figs. 7a,b*, or in chaotic behaviour characterized by phases of almost periodic fluctuations irregularly interrupted by sudden "bursts" of erratic fluctuations (*intermittency*), as in the case represented in *Figs. 7c,d.* It may also happen that such bursts characterize only the transient part of the trajectory, before the system settles down on an attractor with fluctuations of almost constant amplitude (as in the case of *Figs. 7e,f*, obtained with small changes in the parameters of *Figs. 7c,d*).

As shown in the next section, the interaction between the deterministic nonlinear dynamic phenomena of this model and simple exogenous stochastic factors may generate the characteristic features of financial time series, such as *kurtosis*, *skewness* and *fat tails* of return distributions.

Fig. 6

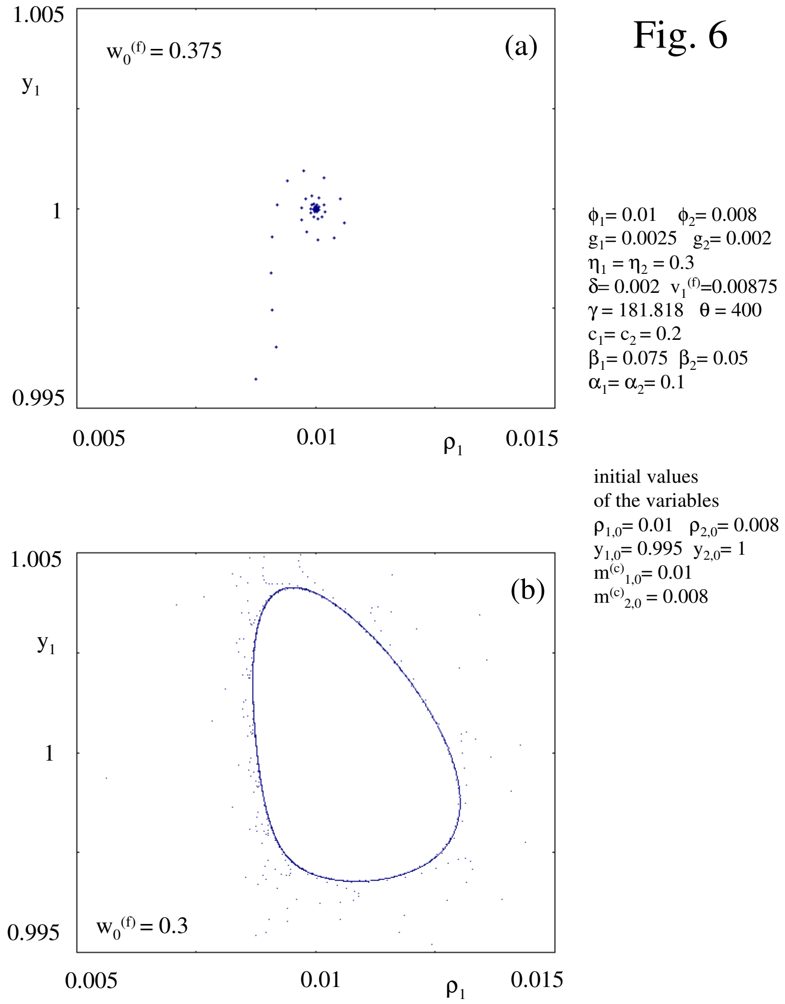


Figure 6: dynamic variables converging to their fundamental levels;

(a) *or fluctuating around the fundamental levels;*

(b) *according to the initial proportion of fundamentalists' wealth.*

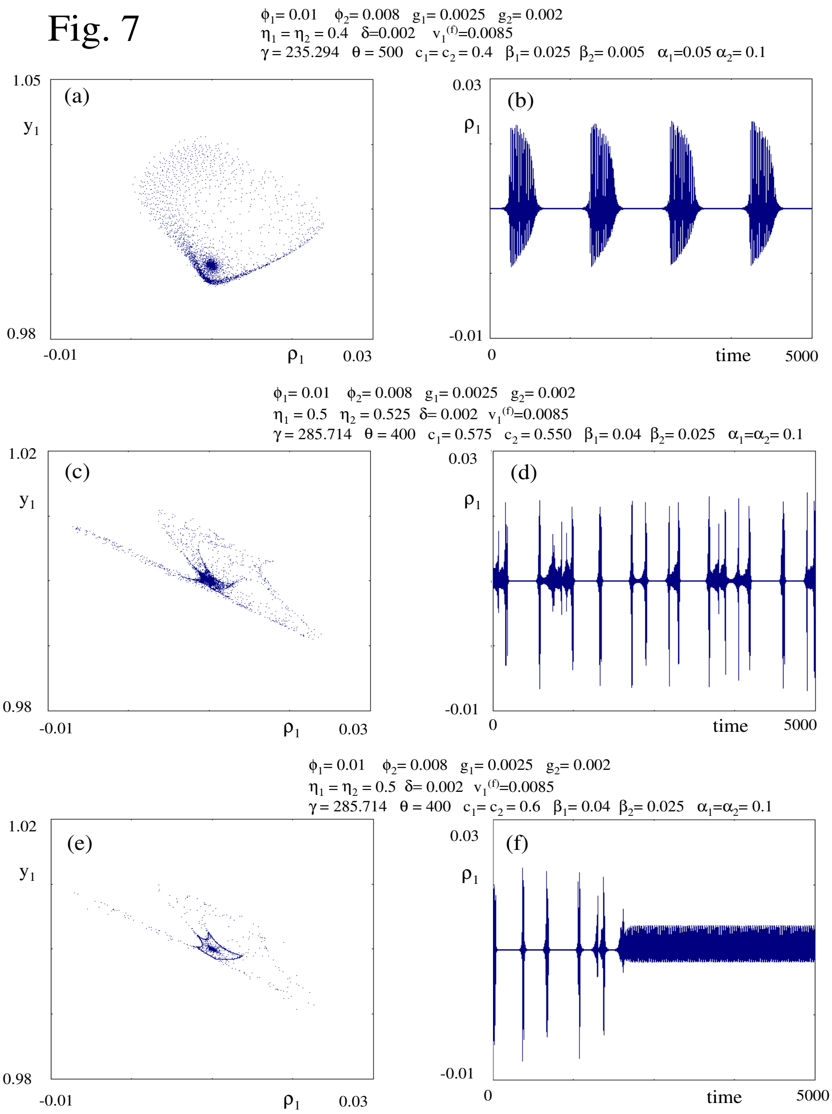


Figure 7: cases of intermittent behavior;

(a),(b) *periodic switching to phases of large fluctuations;*

(c),(d) *chaotic "bursts";*

(e),(f) *intermittency in the transient part of a trajectory.*

7 Stochastic simulations

It is of interest to see how the dynamic features of the nonlinear heterogeneous agents' model established in this paper are affected by simple noise processes. The aim of this section is to show that such an interaction may generate some of the basic characteristics of time series of returns in financial markets such as *fat tails*, *skewness*, *peaked distributions* and *volatility clustering*.

In our examples, stochastic factors are considered by adding a normally distributed noise in the difference equation for the dynamic variable $m^{(c)}_{1,t}$, so that the chartist expected return on the risky asset between t and $t+1$ would be given by

$$m^{(c)}_{1,t} = (1-c_1)m^{(c)}_{1,t-1} + c_1\rho_{1,t} + \xi_t \ ,$$

where the ξ_t are i.i.d. normally distributed random variables with zero mean. This captures the notion that chartists adjust their estimate of the risky return over the next period according to randomly arriving good or bad news in the market. The dynamic equation for $m^{(c)}_1$ of the map (14) would thus be rewritten as:

$$m^{(c)\prime}_1 = (1-c_1)m^{(c)}_1 + c_1\rho'_1 + \xi \qquad \xi \sim \mathcal{N}(0,\sigma^2).$$

Figs. 8a,b display the time series of returns on asset 1 and the return distribution (compared with the corresponding normal distribution) resulting from a simulation of (14) with a normally distributed noise with $\sigma = 0.0006124$. The parameters are the one relevant for the chaotic regime illustrated in *Fig. 5b.* The stochastic version of the model is able to generate fat tails, skewness and kurtosis.

Fig. 8c,d display the time series of returns and the return distribution obtained adding a normally distributed noise ($\sigma = 0.0006532$) to the deterministic situation of intermittency shown in *Figs. 7c,d.* Again the stochastic model clearly shows the distributional characteristics of real financial data. Similar features are present in *8e,f* ($\sigma = 0.0002041$), associated to the deterministic trajectories represented in *Figs. 7e,f.* In particular, the phenomenon of volatility clustering observed in the latter cases is related to the underlying deterministic switching between phases with fluctuations of different amplitude.

Similar features, in particular volatility clustering, may be obtained in the cases of coexistence of attracting sets (as the one already illustrated in the previous section), by allowing exogenous stochastic shocks to move the phase-point between the basins of the coexisting asymptotic states.

It is worth noting how such distributional characteristics do not appear if we allow normal random disturbances when the deterministic dynamics exhibit a monotonic convergence to steady state, as in the case of *Fig. 9* ($\sigma = 0.001306$). In this case the distribution of the capital gain ρ_1 is approximately normal.

Of course such simulations are not expected to mimic the real data, but they have been chosen to illustrate qualitatively how non normally distributed returns similar to the ones observed in financial markets may be the result of the interaction of nonlinear deterministic dynamic phenomena, such as chaotic fluctuations, intermittency or coexistence of attracting sets, shocked by normally distributed noise.

Fig. 8

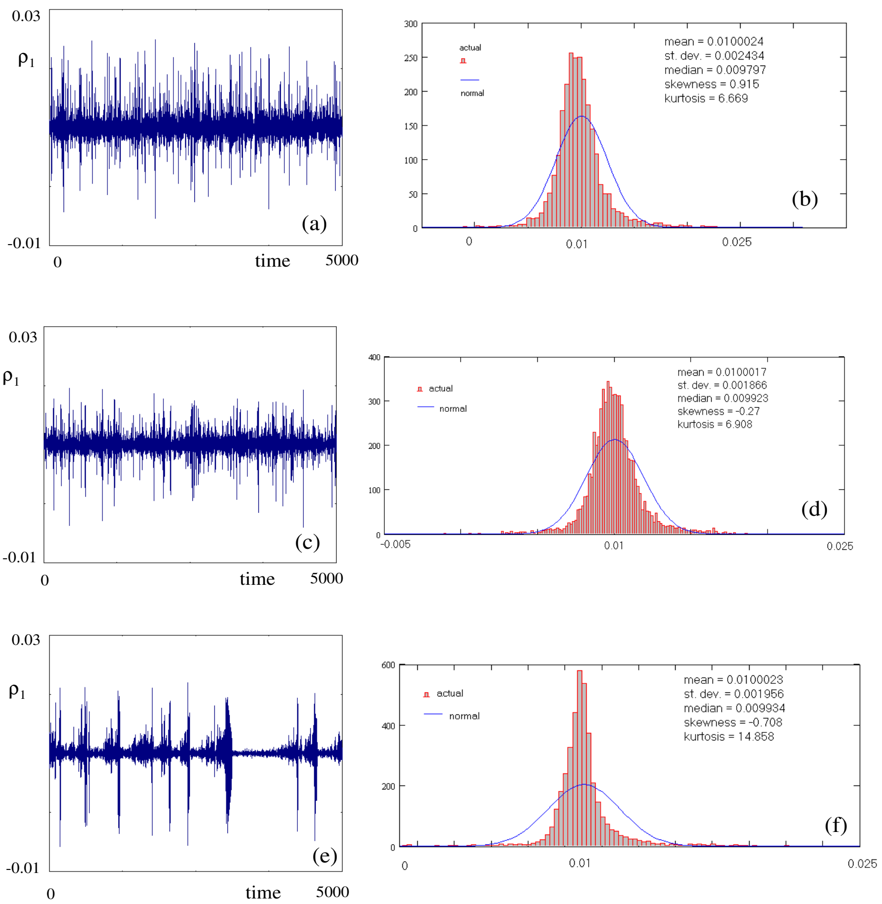


Figure 8: effect of a normally distributed shock affecting the chartists' expected return $m_1^{(c)}$;

(a),(b) *stochastic time series and distribution of the capital gain ρ_1 in the case of convergence to a chaotic attractor;*

(c),(d) and (e),(f) *stochastic time series and distribution of ρ_1 in the case of irregular chaotic bursts.*

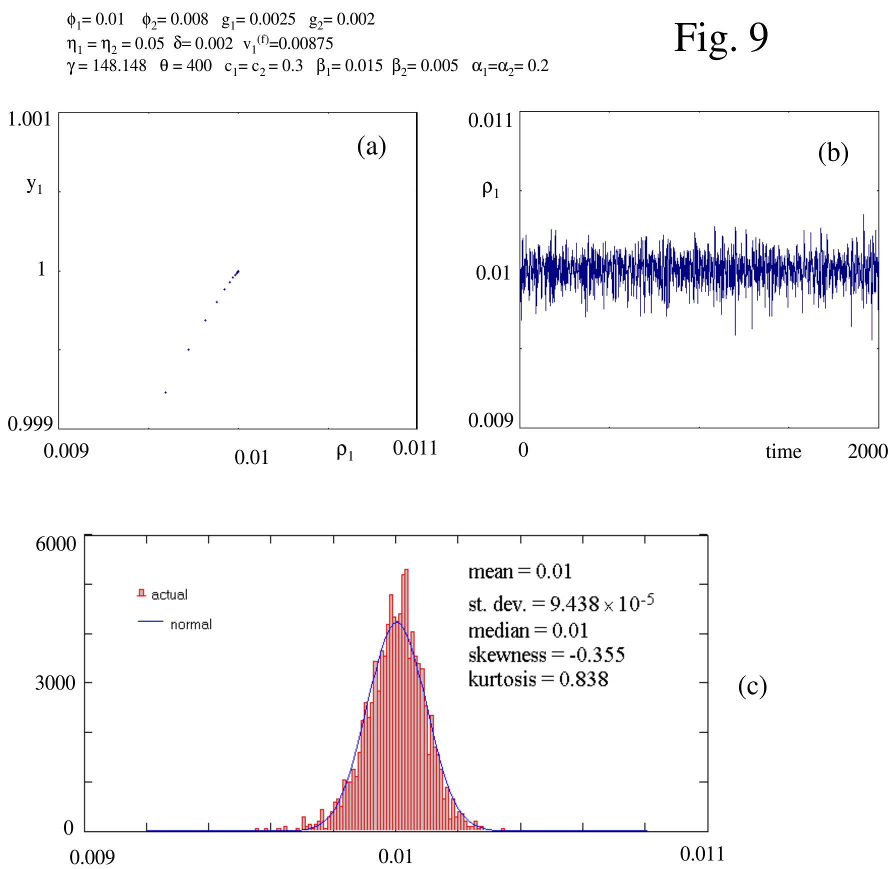


Figure 9: effect of a normally distributed shock affecting the chartists' expected return $m_1^{(c)}$;

(a) *deterministic trajectory;*

(b),(c) *stochastic time series and distribution of the capital gain ρ_1 in a case where the dynamic variables converge monotonically to their fundamental levels.*

8 Conclusions

In the present paper we have set up a model of heterogenous agents (fundamentalists and chartists) investing in a portfolio of a risky asset and a risk-free asset. Each group forms expectations about asset returns and allocates its wealth between the two assets according to one-period expected utility maximization. The investors differ with respect to their "beliefs" about the conditional expected returns of the two assets and the variance of the risky return. Market clearing is effected by a market maker whose price adjustment rules take account of agents' excess demand but also seek to maintain asset prices close to their exogenously determined fundamental values in the long-run. Due to the assumed CRRA utility functions, investors' optimal decisions depend on their wealth and this results in growing prices and wealth.

We set up the high-dimensional dynamical system arising from the interaction and dynamic updating of beliefs of the various agents across the markets for the two assets, and we reduce it to a stationary system where the dynamic variables are actual and expected returns, ratios between fundamental values and prices, and wealth proportions of the two groups of agents. Using both analytical and numerical techniques, we are able to characterize the steady states of the model in this case, as well as other invariant sets on which the dynamics are described by lower dimensional maps.

We then focus on the out-of-equilibrium dynamics of the model: here analytical results seem difficult, so that we mainly use numerical simulations to study the out-of-equilibrium behavior. The main characteristics are phenomena of coexistence of attracting sets, phenomena of intermittent behaviour as well as other phenomena of chaotic dynamics: these latter phenomena seem to emerge mainly when the chartists' demand function is sufficiently sloped or when they update sufficiently fast their expectations. Stochastic simulations are also performed in order to show how the interaction of the nonlinear dynamic phenomena of this model with simple noise processes can give rise to the qualitative types of behaviour observed in real financial data.

It still remains to undertake a more thorough numerical study of the effect of changes of key parameters of the model, such as the ones characterizing the agents' demand functions and the market maker's price adjustment rule parameters, as well as the impact of exogenous stochastic factors. This kind of analysis will require an interplay among theoretical and numerical methods, which is typical for the study of the global dynamic properties of nonlinear dynamical systems of dimension greater than one, as stressed in

Mira et al. (1996) and Brock and Hommes (1997).

Appendix

One period intertemporal optimization with two assets

The expected utility maximization problem faced by agent j at time t is the following (we drop the superscript j):

$$\max_{Z_{1,t}} E_t[U(\Omega_{t+1})] \ . \tag{17}$$

The time evolution of the utility of wealth $U(\Omega)$ is modelled by assuming that the wealth follows a continuous time stochastic differential equation of the type

$$d\Omega = a(\Omega, t)dt + \sum_{i=1}^{2} b_i(\Omega, t)dz_i(t), \tag{18}$$

where z_i, $i = 1, 2$, are standard uncorrelated Wiener processes. Then from Ito's lemma the utility of wealth $X = U(\Omega)$ evolves according to the continuous time stochastic differential equation

$$dX = \mu(X, t)dt + \sum_{i=1}^{2} \sigma_i(X, t)dz_i(t), \tag{19}$$

where

$$\mu(X,t) = U'(U^{-1}(X))a(U^{-1}(X), t) + \frac{1}{2}U''(U^{-1}(X))\sum_{i=1}^{2} b_i^2(U^{-1}(X), t) \ ,$$

$$\sigma_i(X, t) = U'(U^{-1}(X))b_i(U^{-1}(X), t), \qquad i = 1, 2,$$

and $\Omega = U^{-1}(X)$ is the inverse function of the utility of wealth.

Using the Euler-Maruyama discretization one obtains the following discrete-time approximations for (18) and (19):

$$\Omega_{t+\Delta t} - \Omega_t = a_t(\Omega_t)\Delta t + \sum_{i=1}^{2} b_{i,t}(\Omega_t)\Delta z_{i,t}, \tag{20}$$

$$X_{t+\Delta t} - X_t = \mu_t(X_t)\Delta t + \sum_{i=1}^{2} \sigma_{i,t}(X_t)\Delta z_{i,t}, \tag{21}$$

with $\Delta z_{i,t} \sim \mathcal{N}(0, \Delta t)$, $i = 1, 2$. In particular, from (21) one obtains

$$E_t[X_{t+\Delta t}] = X_t + \mu_t(X_t)\Delta t \; . \tag{22}$$

Rescaling the time unit, equation (22) becomes:

$$E_t[X_{t+1}] = X_t + \mu_t(X_t),$$

i.e.:

$$E_t[U(\Omega_{t+1})] = U(\Omega_t) + U'(\Omega_t)a_t(\Omega_t) + \frac{1}{2}U''(\Omega_t)\sum_{i=1}^{2} b_{i,t}^2(\Omega_t), \tag{23}$$

while equation (20) may be rewritten as

$$\Omega_{t+1} = \Omega_t + a_t(\Omega_t) + \sum_{i=1}^{2} b_{i,t}(\Omega_t)\varepsilon_{i,t}, \tag{24}$$

where the $\varepsilon_{i,t}$, $i = 1, 2$, are $\mathcal{N}(0, 1)$ processes.

In order to specify the coefficients $a_t(\Omega_t)$ and $b_t(\Omega_t)$ in a way consistent with our model, we recall that, from eq. (4) the time evolution of wealth of agent j is given by:

$$\Omega_{t+1} = \Omega_t + \Omega_t Z_{1,t}(\rho_{1,t+1} + g_1) + \Omega_t(1 - Z_{1,t})(\rho_{2,t+1} + g_2) \; . \tag{25}$$

Denoting by $m_{i,t}$ and $V_{i,t}$ the belief of agent j at time t about expectation and variance of $\rho_{i,t+1}$, $i = 1, 2$, we assume that

$$\rho_{i,t+1} = m_{i,t} + \sqrt{V_{i,t}}\; \varepsilon_{i,t} \qquad i = 1, 2$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are $\mathcal{N}(0, 1)$ processes, with $E_t(\varepsilon_{1,t}\, \varepsilon_{2,t}) = 0$. Substituting into (25) one easily obtains

$$\begin{aligned} \Omega_{t+1} \;\; = \;\; & \Omega_t + \Omega_t[Z_{1,t}(m_{1,t} + g_1) + (1 - Z_{1,t})(m_{2,t} + g_2)] + \\ & + \Omega_t Z_{1,t}\sqrt{V_{1,t}}\,\varepsilon_{1,t} + \Omega_t(1 - Z_{1,t})\sqrt{V_{2,t}}\,\varepsilon_{2,t} \; , \end{aligned} \tag{26}$$

It follows that the coefficients $a_t(\Omega_t)$ and $b_{i,t}(\Omega_t)$, $i = 1, 2$, in eq. (24) can be consistently specified as

$$a_t(\Omega_t) = \Omega_t[Z_{1,t}(m_{1,t} + g_1) + (1 - Z_{1,t})(m_{2,t} + g_2)] \; ,$$

$$b_{1,t}(\Omega_t) = \Omega_t Z_{1,t}\sqrt{V_{1,t}} \; ; \qquad b_{2,t}(\Omega_t) = \Omega_t(1 - Z_{1,t})\sqrt{V_{2,t}} \; ,$$

and thus the expected utility of wealth $E_t[U(\Omega_{t+1})]$ is given by:

$$\begin{aligned} E_t[U(\Omega_{t+1})] \ &\approx \ U(\Omega_t) + \Omega_t U'(\Omega_t)[Z_{1,t}(m_{1,t}+g_1) + (1-Z_{1,t})(m_{2,t}+g_2)] + \\ &\quad + \frac{1}{2}\Omega_t^2 U''(\Omega_t)[Z_{1,t}^2 V_{1,t} + (1-Z_{1,t})^2 V_{2,t}] \ . \end{aligned}$$

Thus the first order condition of the problem (17) leads to the (approximate) optimum solution

$$Z_{1,t} = -\frac{U'(\Omega_t)}{\Omega_t U''(\Omega_t)} \frac{(m_{1,t}+g_1)-(m_{2,t}+g_2)}{V_{1,t}+V_{2,t}} + \frac{V_{2,t}}{V_{1,t}+V_{2,t}} \ .$$

In the particular case of logarithmic utility, $U(\Omega) = \log \Omega$, one obtains:

$$Z_{1,t} = \frac{(m_{1,t}+g_1)-(m_{2,t}+g_2)+V_{2,t}}{V_{1,t}+V_{2,t}} \ . \tag{27}$$

Throughout this paper we assume that asset 2 (a bond) is a "safe" asset in agents' beliefs (i.e., we take $V_{2,t}^{(j)} = 0$, $j \in (f,c)$), although we allow agents to adjust over time their estimate of the conditional expected return $m_{2,t}^{(j)}$. In this case eq. (27) reduces to the form

$$Z_{1,t} = \frac{(m_{1,t}+g_1)-(m_{2,t}+g_2)}{V_{1,t}}, \tag{28}$$

which is the one we use. We may also consider eq. (28) as an approximation of the optimal investment rule (27) for the case where $V_{2,t}$ is strictly positive but sufficiently small as compared with $V_{1,t}$.

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